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On Multi-Parameter Bifurcation in Diffusion Models

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The problem of bifurcation of periodic orbits from equilibrium when several parameters are present is discussed. The theory is developed from the viewpoint of differential equations on function spaces using the center manifold theory and the method of averaging. Theoretical and numerical analysis of a reaction-diffusion model is included.

1. INTRODUCTION

Diffusion and reaction-diffusion models with nonlinear boundary conditions have been studied by many authors. For example, Thames and Elster [7], Thames and Aronson [6] and Turner and Ames [8] have investigated the problem of coupled enzymes imbedded in membrane walls. Mann and Wolf [5] encountered a diffusion model in the study of heat transfer between solids and gasses and Levinson [4] discussed a similar model in connection with superfluidity.

Under certain conditions these models exhibit oscillatory and periodic behavior. In this paper we show how bifurcation theory can be employed to explain these phenomena when several parameters are present in the model. The existence of Hopf bifurcation in a two parameter reaction-diffusion equation has been pointed out by Thames and Aronson [6] and Chow and White [3]. Much of the theory developed in this work was motivated by the study of this model with three parameters.

The natural phase space for the equations in these and other physical models is an infinite dimensional function space, the nature of which depends on the type of problem. In order to develop the theory, we cnsider a

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multi-parameter ordinary differential equation involving infinite dimensional variables.

Let Y be a Banach space with norm $|\cdot|$ and $F(\alpha, \cdot)$ a smooth nonlinear map of a subspace X of Y into Y depending on a parameter $\alpha \in R^n$. We consider the differential equation

$$\dot{z} = F(\alpha, z), \qquad \cdot = d/dt, \quad z \in X. \tag{1.1}$$

Assume that $F(\alpha, 0) = 0$ so that if $|z|$ is small enough, we have

$$F(\alpha, z) = A(\alpha)\, z + H(\alpha, z),$$

where $A(\alpha) = F_z(\alpha, 0)$ and $|H(\alpha, z)| = O(|z|^2)$ uniformly in α as $|z| \to 0$.

Suppose there exists a smooth manifold $\mathcal{O} \subset R^n$ of codimension one (by smooth we will mean that $\mathcal{O}$ has a tangent space at each $\alpha \in \mathcal{O}$) so that if $\alpha^* \in \mathcal{O}$ then $A(\alpha^*)$ has a single pair of simple, nonzero, pure imaginary eigenvalues $\pm i\omega(\alpha^*)$. Further, assume that the remainder of the spectrum of $A(\alpha)$ remains a uniform positive distance away from the imaginary axis for α in some uniform neighborhood of $\mathcal{O}$.

Fix $\alpha^* \in \mathcal{O}$ and let p^* be transversal to the tangent space to $\mathcal{O}$ at α^*. Let $\alpha = \alpha^* + \beta p^*$ and define $A_{\alpha^*}(\beta) = A(\alpha)$ for $|\beta|$ small. Then $A_{\alpha^*}(\beta)$ has eigenvalues $\lambda(\beta) \pm i\omega(\alpha)$ with $\lambda(0) = 0$, $\omega(\alpha^*) \neq 0$. We assume that $\lambda'(0) \neq 0$.

Under the above assumptions (1.1) will exhibit a Hopf bifurcation, that is, (1.1) has a nonconstant periodic solution $z(t, \beta)$ of period $2\pi\omega(\alpha^*)^{-1} + O(|\beta|^{1/2})$ so that $z(t, \beta) \to 0$ as $\beta \to 0$. Whether the periodic orbits exist for $\beta > 0$ (supercritical bifurcation) or for $\beta < 0$ (subcritical bifurcation) is determined by the nonlinearity $H(\alpha, z)$ and the sign of $\lambda'(0)$. Since these quantities depend on $\alpha^* \in \mathcal{O}$ one may expect that $\mathcal{O}$ can be partitioned into regions $\mathcal{O}_1, \mathcal{O}_2, \ldots$, where $\mathcal{O} = \overline{\bigcup \mathcal{O}_j}$ and the bifurcation is of constant direction (supercritical or subcritical) on each $\mathcal{O}_j$.

2. Decomposition and the Center Manifold

Let $P = P(\alpha) = P(\alpha^* + \beta p^*)$ denote the two dimensional eigenspace corresponding to $\lambda(\beta) \pm i\omega(\alpha)$ and $Q = Q(\alpha)$ its compliment in Y. Then $Y = P \oplus Q$, $X = P \oplus \hat{Q}$, where $\hat{Q} = X \cap Q$, thus any $z \in X$ can be written as $z = (x, y)$, where $x \in P$, $y \in \hat{Q}$. Furthermore, $A_{\alpha^*}(\beta)\, z = (A_P(\beta)\, x,\ A_Q(\beta)\, y)$ where $A_P(\beta) = A_{\alpha^*}(\beta)|_P$ and $A_Q(\beta) = A_{\alpha^*}(\beta)|_{\hat{Q}}$. Since $\lambda'(0) \neq 0$ we can reparameterize so as to have $\lambda(\beta) = \beta$ and by choosing the correct basis for P we can assume that $A_P(\beta)$ has the following matrix representation:

$$A_P(\beta) = \begin{pmatrix} \beta & -\omega(\alpha) \\ \omega(\alpha) & \beta \end{pmatrix}.$$

Here and in the sequel we do not distinguish between $A_P(\alpha)$, x and their representations relative to this basis.

Under this decomposition (1.1) becomes

$$\begin{aligned} \dot{x} &= A_P(\beta)\,x + f(\alpha, x, y), \\ \dot{y} &= A_Q(\beta)\,y + g(\alpha, x, y), \end{aligned} \tag{2.1}$$

where $f(\alpha, x, y) = H(\alpha^* + \beta p^*, (x, y))|_P$ and $|f| = O((|x| + |y|)^2)$ with similar expressions for $g(\alpha, x, y)$.

If $g(\alpha, x, 0) = 0$ then P is an invariant manifold for (2.1) on which the bifurcating periodic orbit must lie. However, one will generically not have this situation. Instead there is a smooth, invariant, two dimensional manifold, $\mathcal{M}$, the Center Manifold, which exists for $|(x, y)|$ and $|\beta|$ small. In addition, $\mathcal{M}$ is tangent to P at $(0, 0)$. Let $\mathcal{M}$ be parametrized by $y = L(\alpha, x)$, then on $\mathcal{M}$ (2.1) becomes the two dimensional equation:

$$\dot{x} = A_P(\beta)\,x + f(\alpha, x, L(\alpha, x)). \tag{2.2}$$

One can now examine the bifurcation behavior of this two dimensional equation in order to determine $\mathcal{O}\!\ell$ and its submanifolds $\mathcal{O}\!\ell_j$.

The basic assumption to guarantee the existance of $\mathcal{M}$ is that the nonlinear semigroup $T(t, z, \alpha)$ in X generated by (1.1) is smooth in z and α for each fixed t.

3. The Two Dimensional Problem

First expand $f(\alpha, x, y)$:

$$\begin{aligned} f(\alpha, x, y) = {} & f_{2,0}(\alpha)\,x^2 + f_{1,1}(\alpha)\,xy + f_{0,2}(\alpha)\,y^2 + f_{3,0}(\alpha)\,x^3 \\ & + f_{2,1}(\alpha)\,x^2y + f_{1,2}(\alpha)\,xy^2 + f_{0,3}(\alpha)\,y^3 + \cdots, \end{aligned}$$

where $f_{j,k}(\alpha)$ are multilinear maps that are sooth in α so that

$$f_{j,k}(\alpha)\colon (\underbrace{R^2 \times \cdots \times R^2}_{j\,\text{times}}) \times (\underbrace{X \times \cdots \times X}_{k\,\text{times}}) \to R^2$$

and we define

$$f_{j,k}(\alpha)\,x^j y^k = f_{j,k}(\alpha)(\underbrace{x, \ldots, x}_{j\,\text{times}}; \underbrace{y, \ldots, y}_{k\,\text{times}}). \tag{3.1}$$

Since we are restricting y to $\mathcal{M}$, we have $y = L(\alpha, x)$ and since $\mathcal{M}$ is tangent to P at $(x, y) = (0, 0)$, we have $L(\alpha, 0) = 0$ and $L_x(\alpha, 0) = 0$. Thus when L is expanded in powers of x we obtain

$$y = L_2(\alpha)\, x^2 + L_3(\alpha)\, x^3 + \cdots,$$

where $L_k(\alpha)\, x^k \in X$ is defined analogously to (3.1).

Under these expansions, (2.2) becomes

$$\dot{x} = A_P(\beta)\, x + G_2(\alpha)\, x^2 + G_3(\alpha)\, x^3 + G_4(\alpha)\, x^4 + G_5(\alpha)\, x^5 + \cdots, \quad (3.2)$$

where

$$G_2(\alpha)\, x = f_{2,0}(\alpha)\, x^2,$$
$$G_3(\alpha)\, x^3 = f_{3,0}(\alpha)\, x^3 + f_{1,1}(\alpha)\, xL_2(\alpha)\, x^2,$$

with similar but more complicated expressions for G_4, G_5, etc.

Convert to polar coordinates by setting $x = (r\cos\theta, r\sin\theta)^T \stackrel{\text{def}}{=} rN_\theta$ (^{T} denotes transpose) and note that

$$\dot{r} = N_\theta^T \dot{x}, \qquad r\dot{\theta} = T_\theta^T \dot{x},$$

where $T_\theta = (-\sin\theta, \cos\theta)^T$. In these coordinates (3.2) becomes

$$\begin{aligned} \dot{r} &= \beta r + C_3 r^2 + C_4 r^3 + \cdots + C_{2m+2} r^{2m+1} + O(r^{2m+2}) \\ \dot{\theta} &= \omega(\alpha) + D_3 r + O(r^2), \end{aligned} \quad (3.3)$$

where $C_k = C_k(\alpha, \theta) = N_\theta^T G_{k-1}(\alpha)\, N_\theta^{k-1}$ and $D_k = D_k(\alpha, \theta) = T_\theta^T G_{k-1}(\alpha)\, N_\theta^{k-1}$. Note that C_k, D_k are homomogenous trigonometric polynomials of degree k with coefficients that depend on α.

We now reduce (3.3) to normal form via the method of averaging, that is, introduce the change of coordinates

$$\bar{r} = r + \sum_{k=2}^{2m+1} U_k(\alpha, \theta)\, r^k, \quad (3.4)$$

where $U_k(\alpha, \theta)$ are appropriately chosen trigonometric polynomials of degree $k+1$ to replace the coefficient of r^k in the first equation of (3.3) by its mean value. Since homogeneous trigonometric polynomials of odd degree have mean value zero, the normal form of (3.3) is

$$\begin{aligned} \dot{r} &= \beta r + K_1(\alpha)\, r^3 + K_2(\alpha)\, r^5 + \cdots + K_m(\alpha)\, r^{2m+1} + O(r^{2m+2}), \\ \dot{\theta} &= \omega(\alpha) + O(r), \end{aligned} \quad (3.5)$$

where we have dropped the bar on r. One explicitly finds that

$$K_1(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} C_4(\alpha, \theta) - \frac{1}{\omega(\alpha)} C_3(\alpha, \theta)\, D_3(\alpha, \theta)\, d\theta \tag{3.6}$$

with similar but much more complex expressions for $K_2(\alpha),..., K_m(\alpha)$.

It should be observed at this point that since we are working in a small neighborhood, $|r| \ll 1$, of $x = 0$ that (3.4) is an invertible, "near identity" transformation. For a detailed discussion of this change of coordinates we refer the reader to [1, 2, 9].

Since $\alpha = \alpha^* + \beta p^*$ and $|\beta|$ is small, we may expand $K_1(\alpha) = K_1(\alpha^*) + O(\beta)$ so that if $K_1(\alpha^*) \neq 0$ any periodic orbit that bifurcates from the equilibrium at $\beta = 0$ should have amplitude approximately $r_0 = (-K_1(\alpha^*)^{-1}\beta)^{1/2}$. This is due to the fact that terms of order $O(r^5)$ and $O(\beta r^2)$ in (3.5) are of higher order than the terms in $\beta r + K_1(\alpha^*)\, r^3$. In fact, an application of he implicit function theorem shows that (3.5) has a periodic orbit of amplitude $r = r_0 + O(|\beta|)$ with period $2\pi\omega(\alpha^*)^{-1} + O(|\beta|^{1/2})$. It is clear from the form of r_0 that the bifurcation is supercritical if $K_1(\alpha^*) < 0$ and subcritical if $K_1(\alpha^*) > 0$. Thus the manifold defined by $K(\alpha^*) = 0$ forms the boundary of those regions on which the bifurcation is of constant direction.

Assume $\mathscr{C}_1$: $K_1(\alpha) = 0$ is a smooth manifold of codimension one in R^n that intersects $\mathscr{O}$ transversally along the manifold $\mathscr{B}_1 = \mathscr{O} \cap \mathscr{C}_1$. Then if q^* is in the tangent space to $\mathscr{O}$ at $\alpha^* \in \mathscr{B}_1$, $\{q^*, p^*\}$ spans a two dimensional space transversal to $\mathscr{B}_1$ at α^*.

Set $\alpha = \alpha^* + \beta p^* + \gamma q^*$ for $|\beta| + |\gamma|$ small and expand $K_1(\alpha) = K_1'(\alpha^*)\, q^*\gamma + O(\beta) + O(\gamma^2)$, $K_2(\alpha) = K_2(\alpha^*) + O(\beta) + O(\gamma)$. Assume that $K_1'(\alpha^*) \neq 0$ then $K_1'(\alpha^*)\, q^* \neq 0$, hence we can reparametrize to have $K_1'(\alpha^*)\, q^* = 1$. Suppose that $K_2(\alpha^*) \neq 0$ and consider (3.5) which now takes the form

$$\begin{aligned} \dot{r} &= \beta r + \gamma r^3 + K_2(\alpha^*)\, r^5 + \text{h.o.t.}, \\ \dot{\theta} &= \omega(\alpha^*) + O(r) + O(|\beta|), \end{aligned} \tag{3.7}$$

where h.o.t. $= O(r^3\beta) + O(r^3\gamma^2) + O(r^5\gamma) + O(r^6)$.

Periodic orbits are now expected with amplitude near the positive roots of $\beta + \gamma\rho^2 + K_2(\alpha^*)\,\rho^4 = 0$, that is, $2K_2(\alpha^*)\,\rho^2 = -\gamma \pm (\gamma^2 - 4\beta K_2(\alpha^*))^{1/2}$. A careful application of the implicit function theorem (see [2]) shows there is a curve Γ: $\beta = \nu(\gamma)$, where

$$\begin{aligned} \nu(\gamma) &= (4K_2(\alpha^*))^{-1}\,\gamma^2 + O(|\gamma|^{5/2}) \qquad &\text{if } K_2(\alpha^*)\,\gamma < 0 \\ &= 0 &\text{if } K_2(\alpha^*)\,\gamma > 0 \end{aligned}$$

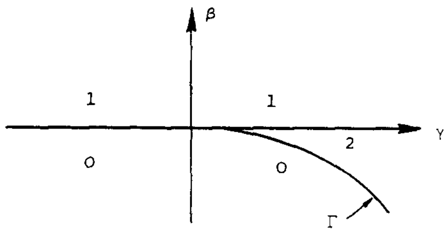

FIG. 1. Parameter space for the case $K_1(\alpha^*) = 0$, $K_1'(\alpha^*) \neq 0$, $K_2(\alpha^*) < 0$. Numerals indicate the number of periodic orbits in each region. Γ is the curve $\beta = \nu(\gamma)$.

so that if (γ, β) is between $\beta = 0$ and Γ then (3.7) has two periodic orbits; one is stable, the other unstable. If $\beta = \nu(\gamma)$ for $K_2(\alpha^*)\,\gamma < 0$ then (3.7) has a unique unstanble periodic orbit (near the equilibrium $r = 0$). The Hopf bifurcation is supercritical if $K_2(\alpha^*)\,\gamma > 0$ and subcritical if $K_2(\alpha^*)\,\gamma < 0$. Figures 1 and 2 illustrate the case $K_2(\alpha^*) < 0$.

The general situation in which $K_1(\alpha^*) = K_2(\alpha^*) = \cdots = K_l(\alpha^*) = 0$, $K_{l+1}(\alpha^*) \neq 0$ can be treated as follows. Let $\{p^*, q_1^*,..., q_l^*\}$ span a space that is transversal to $\mathscr{B}_l = \mathcal{O}\!\mathcal{l} \cap \mathscr{C}_1 \cap \cdots \cap \mathscr{C}_l$, where $\mathscr{C}_j$: $K_j(\alpha) = 0$. Assume q_j^* can be chosen so that $K_i'(\alpha^*)\, q_j^* = \delta(i, j)$ (Kronecker delta). Let $\alpha = \alpha^* + \beta p^* + \gamma_1 q_1^* + \cdots + \gamma_l q_l^*$, then after expanding $K_j(\alpha)$ we consider (3.5) in the form

$$\dot{r} = \beta r + \gamma_1 r^3 + \cdots + \gamma_l r^{2l+1} + K_{l+1}(\alpha^*)\, r^{2l+3} + \text{h.o.t.},$$
$$\dot{\theta} = \omega(\alpha^*) + O(\beta) + O(\mathrm{r}).$$

There are regions in $(\beta, \gamma_1,..., \gamma_l)$-space in which (3.8) has from 0 to $l + 1$ possible periodic orbits. The approximate boundaries of these regions and the number of periodic orbits in each region can be determined by computing the regions in which the right-hand side of the $\dot{r}$ equation of (3.8) has 0 positive roots, 1 positive root, etc. Figures 3 and 4 illustrate the situation when $l = 2$ and $K_3(\alpha^*) < 0$. The details of this computation are not included since it is elementary but quite messy.

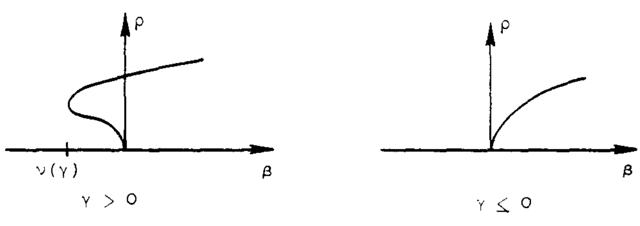

FIG. 2. Bifurcation diagrams for the parameter space in Fig. 1. ρ is the amplitude of the periodic orbit.

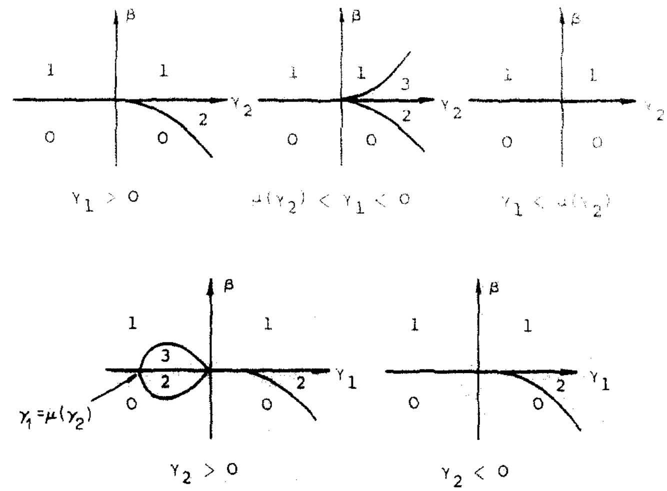

FIG. 3. Parameter space for the vase $l = 2$, $K_3(\alpha^*) < 0$. $\mu(\gamma_2) = O(\gamma_2^2)$.

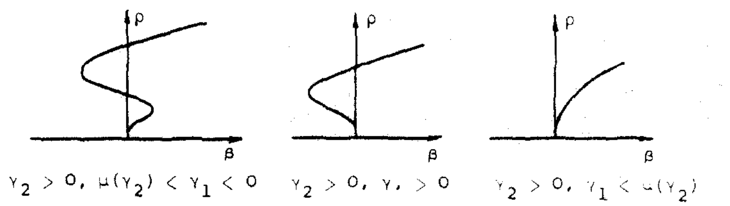

FIG. 4. Sample bifurcation pictures for the parameter space in Fig. 3. ρ is the approximate amplitude of a periodic orbit.

4. A NOTE ON THE COMPUTATION

From (3.2), (3.3) and (3.6) we see that $K_1(\alpha^*)$ depends on the center manifold $\mathcal{M}: y = L(\alpha^*, x)$. However, this dependence is limited to the leading term of L which is $L_2(\alpha^*)x^2$. Close examination shows that the jth averaging constant $K_j(\alpha^*)$ depends on L only through $L^2, L_3, \ldots, L_j$. A procedure by which $L_2(\alpha^*)x^2$ can be computed is given below. A similar method can be employed to compute L_3, L_4, etc.

Let $y = L(\alpha^*, x)$ in (2.1) then after some expansions, one finds that

$$A_Q L_2 x^2 + L_2(x, A_P x) + L_2(A_P x, x) + g_2 x^2 = 0,$$

where $A_Q = A_Q(0)$, $A_P = A_P(0)$ and $g(\alpha^*, x) = g_2 x^2 + O(|x|^3)$. Pass to polar coordinates by setting $x = rN_\theta$ and equate quadratic terms in r to 0 to obtain

$$A_Q L_2 N_\theta^2 + \omega(0) \frac{\partial}{\partial \theta} L_2 N_\theta^2 + g_2 N_\theta^2 = 0. \tag{4.1}$$

We note that $L_2 N_\theta^2$ and $g_2 N_\theta^2$ are real second order homogeneous trigonometric polynomials with coefficients in Y. Let the Fourier expansions of these expressions be given by

$$
\begin{aligned}
g_2 N_\theta^2 &= \sum_{|k|=0,2} g_{2,k} e^{ik\theta}, \qquad g_{2,-k} = \overline{g_{2,k}},\ g_{2,k} \in Y^c, \\
L_2 N_\theta^2 &= \sum_{|k|=0,2} L_{2,k} e^{ik\theta}, \qquad L_{2,-k} = \overline{L_{2,k}},\ L_{2,k} \in X^c,
\end{aligned} \tag{4.2}
$$

where X^c, Y^c are the complexifications of X, Y and bars denote complex conjugate.

Inserting these expansions into (4.1) gives

$$A_Q L_{2,k} + ik\omega(0) L_{2,k} + g_{2,k} = 0, \qquad |k| = 0, 2. \tag{4.3}$$

These equations have a unique solution provided $g_{2,k}$ is in the range of A_Q since A_Q has no pure imaginary eigenvalues.

5. A Reaction-Diffusion Model with Three Parameters

We consider an equation that models the concentrations of substances in a substrate that are produced by two enzymes. Let the enzymes E_1, E_2 be embedded in impervious walls at $x=0$, $x=1$ and produce s_1, s_2, respectively. Assume that these products diffuse and decay through the substrate at equal rates. Further, suppose that s_1 present at $x=1$ activates the production of s_2 and s_2 at $x=0$ inhibits the production of s_1 (activation–inhibitation).

Activation–activation and inhibitation–inhibitation have been studied by other authors, see, for example, [7, 8]. These cases exhibit monotone growth properties as opposed to oscillation which is observed in the activation–inhibitation case.

Specifically consider the following model

$$
\begin{aligned}
& u_t = u_{xx} - q^2 u, && 0 < x < 1, && t \geqslant 0, \\
& v_t = v_{xx} - q^2 v, && 0 < x < 1, && t \geqslant 0, \\
& u_x(1,t) = v_x(0,t) = 0, && && t > 0, \\
& u_x(0,t) + pqf(v(0,t)) = 0, && && t > 0, \\
& v_x(1,t) - \nu pq(1 - f(u(1,t))) = 0, && && t > 0,
\end{aligned} \tag{5.1}
$$

where $u(x,t)$, $v(x,t)$ represent normalized concentrations of s_1, s_2 respectively, p, q, ν are positive constants and $f(u) = u^2/(1+u^2)$.

The equilibrium states of (5.1) are $u^*(x,p,q,\nu) = a \cosh q(1-x)$,

$v^*(x, p, q, \nu) = b \cosh qx$, where $a = pf(b)/\sinh q$ and $b = \nu p(1 - f(a))/\sinh q$. Set $h = u - u^*$, $k = v - v^*$ in (5.1) to obtain

$$\begin{aligned}
&h_t = h_{xx} - q^2 h, && 0 < x < 1, && t \geqslant 0,\\
&k_t = k_{xx} - q^2 k, && 0 < x < 1, && t \geqslant 0,\\
&h_x(1, t) = k_x(0, t) = 0, && && t > 0,\\
&h_x(0, t) + pqf'(b)\, k(0, t) + g_0(p, q, h, k) = 0, && && t > 0,\\
&k_x(1, t) + \nu pqf'(a)\, h(1, t) + g_1(\nu, p, q, h, k) = 0, && && t > 0,
\end{aligned} \tag{5.2}$$

where g_0 and g_1 are the nonlinear functionals defined by

$$g_0(p, q, h, k) = pq(f(v(0, t)) - f'(b)\, k(0, t)),$$
$$g_1(\nu, p, q, h, k) = \nu pq(f(u(1, t)) - f'(a)\, h(1, t) - 1).$$

System (5.2) can be written as an ordinary differential equation in an appropriate Banach space as follows:

$$\dot{V} = A(\nu, p, q)\, V + G(\nu, p, q, V), \tag{5.3}$$

where for each t

$$V = V(t) = \begin{bmatrix} h(\cdot, t) \\ k(\cdot, t) \end{bmatrix} \in H^2 \times H^2,$$

(here and in the sequel H^n denotes the standard Sobelov space on $(0, 1)$) and $A = A(\nu, p, q)$ is defined by $A: X \overset{\text{def}}{=} H^2 \times H^2 \to H^0 \times H^0 \times R^2 \times R^2 \overset{\text{def}}{=} Y$ with

$$AV = (V'' - q^2 V, B_0 V, B_1 V),$$

where

$$B_0 V = \begin{bmatrix} k'(0) \\ h'(0) + pqf'(b)\, k(0) \end{bmatrix},$$

$$B_1 V = \begin{bmatrix} k'(1) + \nu pqf'(a)\, h(1) \\ h'(1) \end{bmatrix},$$

and

$$G(V) = (0, G_0(V), G_1(V)),$$

with

$$G_0(V) = \begin{bmatrix} 0 \\ g_0(p, q, h, k) \end{bmatrix}, \qquad G_1(V) = \begin{bmatrix} g_1(\nu, p, q, h, k) \\ 0 \end{bmatrix}.$$

Identify the domain of A, X, with a nondense subset of Y in the obvious way. If $U=(V,c,d)\in Y$ define $|U|=(|h|_0^2+|k|_0^2+\|c\|^2+\|d\|^2)^{1/2}$, where $|h|_0^2=\langle h,h\rangle$, $\langle h,k\rangle=\int_0^1 h(x)\,k(x)\,dx$ and $\|\cdot\|$ is the Euclidian norm on R^2. We then have $|G(V)|=O(|V|^2)$ uniformly in ν, p, q as $|V|\to 0$.

Let Y^* be the adjoint space of Y and define the bilinear form $[\cdot,\cdot]$ on $Y^*\times Y$ by

$$[(\psi,c_1,d_1),(\varphi,c_2,d_2)]=c_1\cdot\varphi(0)-d_1\cdot\varphi(1)$$
$$+\psi(0)\cdot c_2-\psi(1)\cdot d_2+\langle\psi,\varphi\rangle. \tag{5.4}$$

We will use this form to decompose (5.3). However, we must first discuss the eigenvalues of A.

A straightforward computation shows that the eigenvalues of A are found by solving the following equations in τ and σ

$$\tau\sin\tau\cosh\sigma-\sigma\cos\tau\sinh\sigma=0,$$
$$\sigma\sin\tau\cosh\sigma+\tau\cos\tau\sinh\sigma=pq\sqrt{\nu f'(a)f'(b)}. \tag{5.5}$$

Then A has eigenvalues $\lambda(p,q,\nu)=\sigma^2-\tau^2-q^2\pm 2i\tau\sigma$ with eigenfunctions φ and $\bar{\varphi}$, where

$$\varphi=\begin{pmatrix}\alpha\cos\mu(1-x)\\ \beta\cos\mu x\end{pmatrix}$$

with $\mu=\tau+i\sigma$, $\sigma>0$ and $(\alpha,\beta)^T$ is in the null space of

$$\begin{pmatrix}\mu\sin\mu & pqf'(b)\\ \nu pqf'(a) & -\mu\sin\mu\end{pmatrix},$$

which is singular as seen by (5.5).

Numerical study shows that there exists a two dimensional surface $p=h(q,\nu)$ on which $\operatorname{Re}\lambda(p,q,\nu)=0$ (see Fig. 6). Further $\operatorname{Re}\lambda(p,q,\nu)>0$ for $p>h(q,\nu)$, $\operatorname{Re}\lambda(p,q,\nu)<0$ for $p<h(q,\nu)$, thus Hopf bifurcation is expected as p crosses through this manifold.

Let P denote the two dimensional eigenspace of A with basis $\Phi=(\varphi_2,\varphi_1)$, where $\varphi_1=\frac{1}{2}(\varphi+\bar{\varphi})$ and $\varphi_2=(1/2i)(\varphi-\bar{\varphi})$. Then A restricted to P is represented by the matrix

$$A_P=\begin{pmatrix}\sigma^2-\tau^2-q^2 & -2\tau\sigma\\ 2\tau\sigma & \sigma^2-\tau^2-q^2\end{pmatrix}\overset{\text{def}}{=}\begin{pmatrix}\beta & -\omega\\ \omega & \beta\end{pmatrix}.$$

Let ψ be a dual basis for Φ relative to (5.4) so that $\langle\Psi,\Phi\rangle=I=2\times 2$ identity (A is self adjoint on P). In fact, $\Psi=\langle\Phi^T,\Phi\rangle^{-1}\Phi$. If

$U=(V,c,d)\in Y$ then $U=U_P+U_Q$ where $U_P=(\Phi\langle\Psi,V\rangle,0,0)$ and $U_Q=U-U_P$. Thus if $V(t)\in X\subset Y$ we can write

$$V(t)=\Phi x(t)+y(t),$$

where

$$x(t)=\langle\Psi,V(t)\rangle,$$
$$y(t)=V(t)-\Phi x(t).$$

We decompose (5.3) in P and its compliment in Y using (5.4) to obtain

$$\begin{aligned}\dot{x}(t)&=A_Px(t)+\Psi(0)\,G_0-\Psi(1)\,G_1,\\ \dot{y}(t)&=Ay(t)+(\Phi(\Psi(1)\,G_1-\Psi(0)\,G_0),G_0,G_1).\end{aligned}\tag{5.6}$$

We now pass to polar coordinates in (5.6) by setting $x(t)=rN_\theta$ to obtain

$$\begin{aligned}\dot{r}&=\beta r+N_\theta^T(\Psi(0)\,G_0(\Phi N_\theta r+y)-\Psi(1)\,G_1(\Phi N_\theta r+y)),\\ \dot{\theta}&=\omega+r^{-1}T_\theta^T(\Psi(0)\,G_0(\Phi N_\theta r+y)-\Psi(1)\,G_1(\Phi N_\theta r+y)),\\ \dot{y}&=Ay+(\Phi\{\Psi(1)\,G_1(\Phi N_\theta r+y)-\Psi(0)\,G_0(\Phi N_\theta r+y)\},\\ &\quad G_0(\Phi N_\theta r+y),G_1(\Phi N_\theta r+y)).\end{aligned}\tag{5.7}$$

To compute $K_1(\nu,q)$ we must first determine $L_2(\nu,q)\,N_\theta^2$, which from (4.1) satisfies

$$AL_2N_\theta^2+\omega\frac{\partial}{\partial\theta}L_2N_\theta^2+g_2N_\theta^2=0,$$

where

$$\begin{aligned}g_2N_\theta^2=\Bigg(&\tfrac{1}{2}pq\Phi\left\{\nu f''(a)\,\Psi(1)\begin{pmatrix}\Gamma_1^2(1)\\0\end{pmatrix}-f''(b)\,\Psi(0)\begin{pmatrix}0\\\Gamma_2^2(0)\end{pmatrix}\right\},\\ &\begin{bmatrix}0\\\tfrac{1}{2}pqf''(b)\,\Gamma_2^2(0)\end{bmatrix},\begin{bmatrix}\tfrac{1}{2}\nu pqf''(a)\,\Gamma_1^2(1)\\0\end{bmatrix}\Bigg)\end{aligned}$$

with $(\Gamma_1(x),\Gamma_2(x))^T=\Phi(x)\,N_\theta$. Let the Fourrier expansions of $L_2N_\theta^2$ and $g_2N_\theta^2$ be given by (4.2), then (4.3) becomes

$$\begin{aligned}&(L_{2,k}''-(i\omega k+q^2)\,L_{2,k}+g_{2,k},g_{2,k}(0)+B_0L_{2,k},g_{2,k}(1)+B_1L_{2,k})\\ &\quad=(0,0,0)\qquad\text{for }|k|=0,2,\end{aligned}$$

which is easily solved using variation of constants and imposing the boundary conditions. In fact, $K_1(\nu, q)$ only depends on $L_2(0)$, $L_2'(0)$, $L_2(1)$ and $L_2'(1)$. Once these quantities are known one can compute $K_1(\nu, q)$ using (3.2), (3.3) and (3.6). To evaluate the various integrals that arose in this computation a Romberg scheme was used.

The numerical computation of $K_1(\nu, q)$ on $p = h(\nu, q)$ revealed that $K_1(\nu, q) = 0$ on two disconnected one dimensional manifolds $\mathscr{B}_1$ and $\mathscr{B}_2$ that partition $p = h(\nu, q)$ into three disjoint regions $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{A}_3$ (see Fig. 5). $K_1(\nu, q) < 0$ on $\mathscr{A}_1 \cup \mathscr{A}_3$ and $K_1(\nu, q) > 0$ on $\mathscr{A}_2$. Hence the bifurcation is supercritical on $\mathscr{A}_1 \cup \mathscr{A}_3$ (stable periodic orbit exists for $p > h(\nu, q)$) and subcritical on $\mathscr{A}_2$ (unstable periodic orbit exists for $p < h(\nu, q)$).

If $(\nu, q, \beta) \in \mathscr{A}_2 \times (0, \varepsilon)$ for some small positive ε then a stable periodic orbit that arises in the manner discussed in Section 3 serves as a local attractor since the equilibrium is unstable when $p > h(\nu, q)$. Of course the theory only guarantees this if $(\nu, q) \in \mathscr{A}_2$ is sufficiently close to $\mathscr{B}_1$ or $\mathscr{B}_2$. However, Fig. 5 leads one to conjecture the parameter space shown in Fig. 6. Further numerical study to confirm this conjecture is in progress.

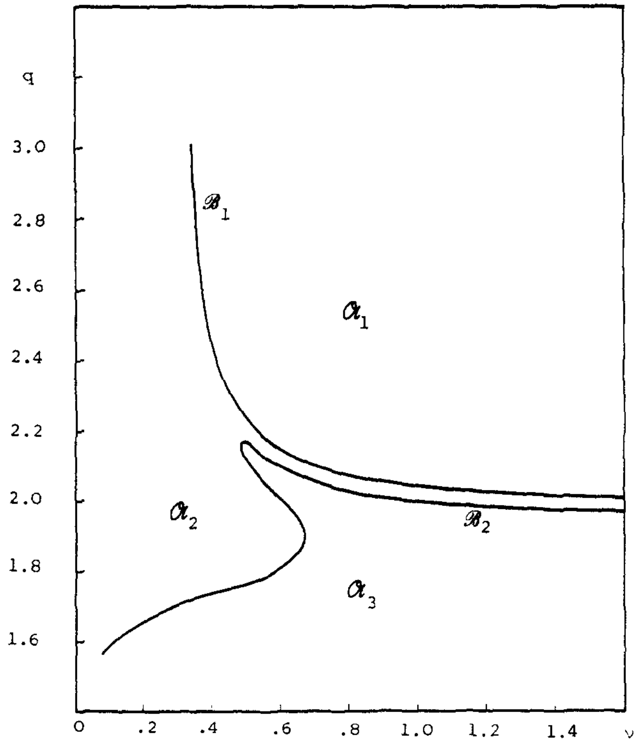

FIG. 5. Parameter space for (5.1) restricted to $p = h(\nu, q)$. Supercritical bifurcation in $\mathscr{A}_1$ and $\mathscr{A}_3$, subcritical bifurcation in $\mathscr{A}_2$. $\mathscr{B}_1$ and $\mathscr{B}_2$ are the curves along which $K_1(\nu, q) = 0$.

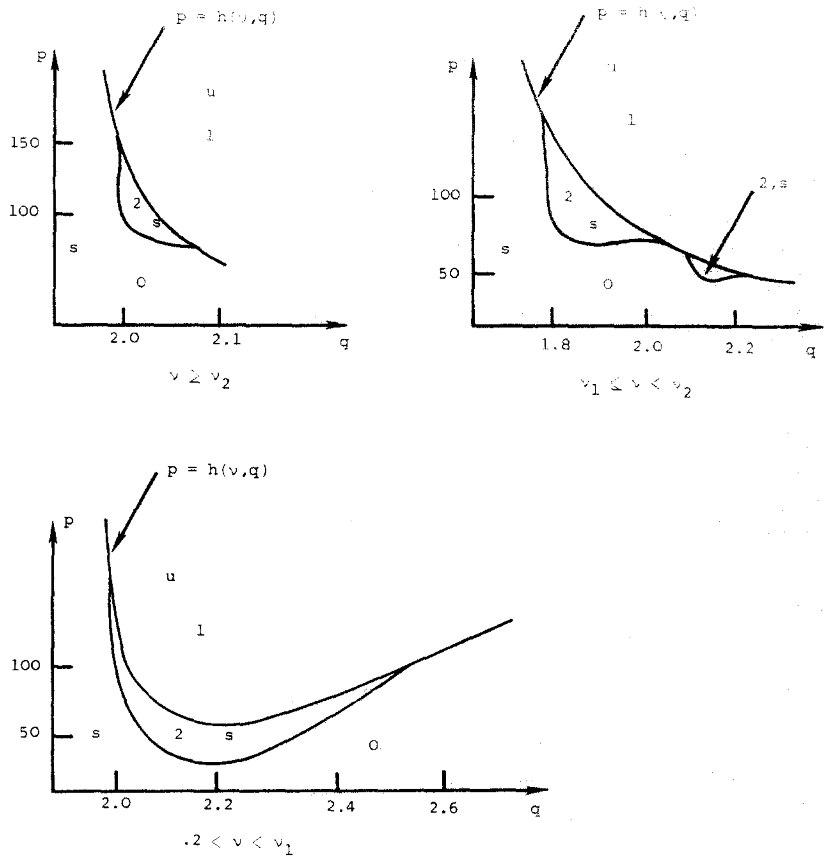

FIG. 6. Conjectured parameter space for (5.1). All curves are approximate. Numerals indicate the number of periodic orbits in each region. $\nu_1 \approx 0.479$, $\nu_2 = 0.667$, $u =$ unstable equilibrium, $s =$ stable equilibrium.

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